# Fuzzy Eigenvalues and Fuzzy Eigen Vectors for Fuzzy Matrix

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**Abstract:**

Fuzzy Eigen Vectors are the extended form of the Real Eigen Vectors of a real Matrix. Some important properties of Fuzzy Eigenvalues and fuzzy eigenvectors of a Fuzzy Matrix are studied.

Key words: Fuzzy number, Fuzzy Matrix, Fuzzy Eigen Vector and Fuzzy Eigenvalue.

## A. Introduction :

The Eigen value problem plays an important role in solving systems of differential equations, analyzing population growth models and calculating powers of matrices. Many important characteristics of physical system, such as, stability, can be determined by knowing the nature of eigenvalues.

Let A be a real matrix of order n. If $\exists \lambda \in \mathbb{R}$ and a non-zero vector $x \in \mathbb{R}^n$ such that

$Ax = \lambda x$ .......... (1)

then $\lambda$ is said to be the real eigenvalue of A and $x$ is said to be the real eigenvector of A corresponding to the real eigenvalue $\lambda$.

Visually, most vectors $x$ will not satisfy the equation (1). When a vector $x$ be multiplied by A then it changes the direction, so that $Ax$ is not a multiple of $x$. It follows that only some special v numbers are eigenvalues and some vectors $x$ are eigenvectors.

## B. Definitions :

In this section we recall some definitions which are used to next section in this paper.

Definition 1: Fuzzy Arithmetic Number of Fuzzy Number (FN): (Ma, 2000, p 55-58)

A fuzzy arithmetic number or simply a fuzzy number is a fuzzy set $\tilde{u} : \mathbb{R} \rightarrow [0,1]$ such that

(i) $\tilde{u}(x)$ is upper semi continuous.
(ii) $\tilde{u}(x) = 0$ outside some closed interval [c,d]
(iii) $\exists s$ two real numbers $a$, b such that $c \leq a \leq b \leq d$ for which
$(a)$ $\tilde{u}(x)$ is monotonic increasing on the closed interval $[c, a]$
$(b)$ $\tilde{u}(x)$ is monotonic decreasing on the closed interval $[b, d]$

and (c) $\tilde{u}(x) = 1\ \forall x \in [a, b]$

The set of all fuzzy arithmetic number is denoted by $\mathfrak{E}$. Obviously the set of all real number $\mathbb{R}$ is a proper subset of$\mathfrak{E}$.

Definition 2: Fuzzy Matrix (FM):

A fuzzy matrix A of order $m \times n$ is defined as $A = [< a_{ij}, a_{ij\mu} >]$ where $a_{ij\mu}$ is the membership value of the element $a_{ij}$ in A. For simplicity, we write A as$A = [a_{ij\mu}]$.

Definition 3: Fuzzy Square Matrix:

An $n \times n$ fuzzy matrix is said to be a fuzzy square matrix of order n.

Definition 4: Orthogonal Fuzzy Matrix:

A fuzzy square matrix $\tilde{A}$ is said to be orthogonal if $\tilde{A}\,\tilde{A}^T = \tilde{A}^T\tilde{A} = I$.

Definition 5: Fuzzy Symmetric Matrix:

A fuzzy square matrix $\tilde{A}$ is said to be symmetric if $\tilde{A} = \tilde{A}^T$.

Definition 6: Similar Fuzzy Matrices:

A fuzzy matrix $\tilde{A}$ is said to be similar to another fuzzy matrix $\tilde{B}$ if there exist a non-singular fuzzy matrix $\tilde{P}$ such that $\tilde{B} = \tilde{P}^{-1}\tilde{A}\tilde{P}$

**C. Fuzzy eigen eigenvalue (FE) and fuzzy eigen vector (FEV) :**

If $\exists \lambda \in \mathbb{R}$ and a non-zero fuzzy vector $\tilde{x} \in \mathbb{R}^n$ satisfying (1) then $\lambda$ is called the real eigenvalue and $\tilde{x}$ is called the fuzzy eigenvector of A corresponding to the real eigenvalue $\lambda$.

Complex eigenvalues and corresponding complex eigenvectors may be occurred for a real matrix. In this paper we mainly interested with real eigenvalues and corresponding fuzzy eigenvectors for a real matrix.

(1) Can be written as a Dual Fuzzy Linear System (Frideman, 2006, p 1257-1275) in the form

$$\underline{\lambda x_i} = \sum_{j=1}^{n} \underline{a_{ij}\, x_j}$$

$$\overline{\lambda x_i} = \sum_{j=1}^{n} \overline{a_{ij}\, x_j} \quad , \quad i = 1, 2, 3, \quad \ldots\ldots \quad , n$$

Where $x_i = (\underline{x_i}(r), \overline{x_i}(r))$

This can be written as

$$\begin{pmatrix} \lambda I_n & 0 \\ 0 & \lambda I_n \end{pmatrix} \begin{pmatrix} \underline{X} \\ -\overline{X} \end{pmatrix} = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_1 \end{pmatrix} \begin{pmatrix} \underline{X} \\ -\overline{X} \end{pmatrix}, \text{ if } \lambda \geq 0 \qquad \ldots\ldots\ldots \qquad (2)$$

$$\text{Or} \quad \begin{pmatrix} 0 & -\lambda I_n \\ -\lambda I_n & 0 \end{pmatrix} \begin{pmatrix} \underline{X} \\ -\overline{X} \end{pmatrix} = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_1 \end{pmatrix} \begin{pmatrix} \underline{X} \\ -\overline{X} \end{pmatrix}, \text{ if } \lambda < 0 \qquad \ldots\ldots\ldots\ldots \qquad (3)$$

Where $X = \left(\underline{x_1}, \underline{x_2}, \quad \ldots\ldots \quad , \underline{x_n}, -\overline{x_1}, -\overline{x_2}, \quad \ldots\ldots \quad , -\overline{x_n}\right)^T = \left(\underline{X}^T, -\overline{X}^T\right)^T$

And $S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_1 \end{pmatrix}$ defined as $a_{ij} \geq 0 \Rightarrow s_{ij} = s_{i+n, j+n} = a_{ij}$

$$a_{ij} < 0 \Rightarrow s_{i+n, j} = s_{i, j+n} = -a_{ij}$$

And any $s_{ij}$ is not determined is zero such that $A = S_1 - S_2$

Here $S_1$ contains non-negative entries of A while $S_2$ contains absolute value of negative entries of A. $S_1 + S_2 = |A| = \left(|a_{ij}|\right)_{n\times n}$. In this paper, $|A|_{m\times n}$ denotes the non-negative matrix $\left(|a_{ij}|\right)_{m\times n}$ where $A = \left(a_{ij}\right)_{m\times n}$.

Lemma: (Tian, 2008, p 161-165)

The homogeneous fuzzy linear system $A\tilde{x} = \hat{0}$ has only real solution vector if A has no column which consists of zero.

Result: (Eigenvalues corresponding to the fuzzy eigenvectors of a real matrix)

Let A be a real matrix. A real number $\lambda$ and a non-zero fuzzy vector $\tilde{x}$ together satisfy the equation (1) only if $\lambda$ is one of eigenvalues of the matrix A or one of the positive eigenvalue of $|A|$ or one of the opposite number of the positive eigenvalue of $|A|$.

Proof:

Case 1: $\lambda$ is one of the eigenvalues of A, i.e, $\det(\lambda I_n - A) = 0$.

In this case the result is obvious.

Case 2: $\det(\lambda I_n - A) \neq 0$

$\Rightarrow \lambda \in \mathbb{R}$ and non-zero fuzzy vector $\tilde{x} = (\widetilde{x_1}, \widetilde{x_2}, \quad \ldots\ldots \quad, \widetilde{x_n})^T$ satisfies (1) only if $\lambda$ and non-zero vector $X = \left(\underline{x_1}, \underline{x_2}, \quad \ldots\ldots \quad, \underline{x_n}, -\overline{x_1}, -\overline{x_2}, \quad \ldots\ldots \quad, -\overline{x_n}\right)^T \in \mathfrak{E}^{2n}[0,1]$ satisfy (2) or (3).

Subcase (i) $\lambda > 0$.

Then, (2) in $\mathfrak{E}^{2n}[0,1]$ has non-zero solution if $\det(\lambda I_{2n} - S) = 0$

$\Rightarrow \det(\lambda I_n - A)\ \det(\lambda I_n - |A|) = 0$

$\Rightarrow \lambda$ is a positive eigenvalue of $|A|$

Subcase (ii) : $\lambda = 0$ is an eigenvalue of $|A|$ but not of A.

In this subcase, $Ax = \hat{0}$ has only trivial solution be the above lemma.

Subcase (iii) : $\lambda < 0$.

Then (3) in $\mathfrak{E}^{2n}[0,1]$ has non-zero solution only if $\det\begin{pmatrix} -s_1 & -\lambda I_n - s_2 \\ -\lambda I_n - s_2 & -s_1 \end{pmatrix} = 0$

$\Rightarrow \det(-\lambda I_n - |A|)\ \det(\lambda I_n - A) = 0$

$\Rightarrow \lambda$ is one of the opposite number of positive eigenvalue of $|A|$.

Note : Below two examples shows that the positive eigenvalues of $|A|$ or their opposite number may also be or not be the real eigenvalues corresponding with fuzzy eigen vectors of A.

Example 1 : Let $A = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$. Here 3 is not a eigenvalues of A.

But $A\tilde{u} = \pm 3\tilde{u}$ with $\tilde{u} = \begin{pmatrix} (0.5r - 0.5 & 0.5 - 0.5r) \\ (0.5r - 0.5 & 0.5 - 0.5r) \end{pmatrix}$. Thus $\tilde{u}$ is a fuzzy eigen vector corresponding to the eigenvalue $\pm 3$ of A.

Example 2 : Let $A = \begin{pmatrix} -5 & 2 \\ 1 & 4 \end{pmatrix}$. $\lambda = \frac{-1 \pm \sqrt{89}}{2}$ are eigenvalues of A. But 3,6 are eigenvalues of $|A|$. Here $A\tilde{u} = \pm 6\tilde{u}$ has non trivial solution in $\mathfrak{E}^n$ and $A\tilde{u} = \pm 3\tilde{u}$ has not.

D. SOME PROPERTIES OF FUZZY EIGENVALUES AND FUZZY EIGEN VECTORS FOR A FUZZY MATRIX :

Property 1:

The sum of the fuzzy eigenvalues equals the sum of the diagonal entries of the fuzzy matrix.

Proof :

Let $\tilde{A}$ be a fuzzy matrix. The characteristic equation of $\tilde{A}$ is $\det(\tilde{A} - \lambda I) = 0$

$\Rightarrow \lambda^n$ - (sum of the diagonal entries of $\tilde{A}$ ) $\lambda^{n-1}$ + $\quad \ldots\ldots \quad + (-1)^n det\tilde{A} = 0$.

Let $\lambda_1, \lambda_2, \quad \ldots\ldots \quad, \lambda_n$ satisfies this equation. Then these are the fuzzy eigenvalues of the fuzzy matrix $\tilde{A}$.

So, $\lambda_1 + \lambda_2 + \quad \ldots\ldots \quad + \lambda_n$ = sum of the diagonal entries of the fuzzy matrix $\tilde{A}$.

Hence the result follows.

Property 2 :

The product of the fuzzy eigenvalues of the fuzzy matrix $\tilde{A}$ equals the determinant of the fuzzy matrix $\tilde{A}$.

Proof :

The characteristic equation of the fuzzy matrix $\tilde{A}$ is $\det(\tilde{A} - \lambda I) = 0$

$\Rightarrow \lambda^n$ - (sum of the diaonal entries of $\tilde{A}$ ) $\lambda^{n-1}$ + ..... + $(-1)^n det\tilde{A} = 0$.

If $\lambda_1, \lambda_2, \quad \ldots\ldots \quad, \lambda_n$ are the fuzzy eigenvalues of the fuzzy matrix $\tilde{A}$ then $\lambda_1 \lambda_2 \quad \ldots\ldots \quad \lambda_n = \det\tilde{A}$

Hence the result follows.

Property 3 :

$\tilde{A}_{n\times n}$ and $\tilde{A}^T{}_{n\times n}$ have the same fuzzy eigenvalues where $\tilde{A}^T{}_{n\times n}$ denotes the transpose of the fuzzy matrix $\tilde{A}_{n\times n}$.

Proof :

The characteristic equation of the fuzzy matrices $\tilde{A}$ and $\tilde{A}^T$ are

$\det(\tilde{A} - \lambda I) = 0$ and $\det(\tilde{A}^T - \lambda I) = 0$ respectively.

Now $\det(\tilde{A} - \lambda I) = \det(\tilde{A} - \lambda I)^T = \det(\tilde{A}^T - \lambda I)$

Hence the result follows.

Property 4 :

The fuzzy eigenvalues of a fuzzy diagonal matrix are its diagonal entries.

Proof :

Let $\tilde{A} = diag(d_1, d_2, \quad \ldots\ldots \quad, d_n)$ be a fuzzy matrix.

Then $\det(\tilde{A} - \lambda I_n) = (d_1 - \lambda)(d_2 - \lambda) \quad \ldots\ldots \quad (d_n - \lambda)$

Hence the roots are $d_1, d_2, \quad \ldots\ldots \quad, d_n$.

So, the fuzzy eigenvalues of the fuzzy matrix $\tilde{A}$ are $d_1, d_2, \quad \ldots\ldots \quad, d_n$.

Hence the result follows.

Property 5 :

If $\lambda(\neq 0)$ is a fuzzy eigenvalue of a fuzzy matrix $\tilde{A}$, then $\frac{1}{\lambda}$ is a eigenvalue of $\tilde{A}^{-1}$.

Proof :

Let X be the eigenvector corresponding to $\lambda$. Then $\tilde{A}X = \lambda X$

$$\Rightarrow \tilde{A}^{-1}(\tilde{A}X) = \tilde{A}^{-1}(\lambda X)$$

$$\Rightarrow (\tilde{A}^{-1}\tilde{A})X = \lambda(\tilde{A}^{-1}X)$$

$$\Rightarrow IX = \lambda(\tilde{A}^{-1}X)$$

$$\Rightarrow \frac{1}{\lambda}X = \tilde{A}^{-1}X$$

$\Rightarrow \frac{1}{\lambda}$ is an fuzzy eigenvalue of the fuzzy inverse matrix $\tilde{A}^{-1}$.

Property 5 :

If $\lambda(\neq 0)$ is a fuzzy eigenvalue of an orthogonal fuzzy matrix $\tilde{A}$, then $\frac{1}{\lambda}$ is also an fuzzy eigenvalue.

Proof :

Since $\tilde{A}$ is an orthogonal fuzzy matrix then $\tilde{A}\,\tilde{A}^T = \tilde{A}^T\tilde{A} = I$ i.e., $\tilde{A}^T = \tilde{A}^{-1}$ ..... (1)

Again $\lambda(\neq 0)$ is an fuzzy eigenvalue of an orthogonal fuzzy matrix $\tilde{A}$ then by previous property $\frac{1}{\lambda}$ is an fuzzy eigenvalue of the fuzzy inverse matrix $\tilde{A}^{-1}$.Using (1) we get $\frac{1}{\lambda}$ is an fuzzy eigenvalue of the fuzzy matrix $\tilde{A}^{T}$. Now $\det(\tilde{A}-\lambda I)=\det(\tilde{A}^{T}-\lambda I)\Rightarrow \tilde{A}$ and $\tilde{A}^{T}$ have the same fuzzy eigenvalues. Hence $\frac{1}{\lambda}$ is also an fuzzy eigenvalue of $\tilde{A}$.

Property 6 :

The similar fuzzy matrices have same fuzzy eigenvalues.

Proof :

Let $\tilde{A}$ and $\tilde{B}$ be two similar fuzzy matrices. Then there exist a non-singular fuzzy matrix $\tilde{P}$ such that $\tilde{B}=\tilde{P}^{-1}\tilde{A}\tilde{P}$.

Now $det(\tilde{B}-\lambda I)=det(\tilde{P}^{-1}\tilde{A}\tilde{P}-\lambda I)=det(\tilde{P}^{-1}\tilde{A}\tilde{P}-\tilde{P}^{-1}\lambda I\tilde{P})=det(\tilde{P}^{-1}(\tilde{A}-\lambda I)\tilde{P})=det(\tilde{P}^{-1})det(\tilde{A}-\lambda I \det P= det A-\lambda I \det P-1 P= det A-\lambda I \det I=det A-\lambda I$.

Thus $\tilde{A}$ and $\tilde{B}$ have the same characteristic polynomial and hence characteristic roots. Therefore they have same fuzzy eigenvalues.

Property 7 :

The fuzzy eigenvector $\tilde{X}$ of an fuzzy matrix $\tilde{A}$ is not unique.

Proof :

Let $\tilde{X}$ be the fuzzy eigenvector corresponding to the fuzzy eigenvalue $\lambda$ for a fuzzy matrix $\tilde{A}$. Then $\tilde{A}\tilde{X}=\lambda\tilde{X}$

$\Rightarrow k(\tilde{A}\tilde{X})=k(\lambda\tilde{X})$ [For a non-zero scalar k]

$\Rightarrow \tilde{A}(k\tilde{X})=\lambda(kX)$

$\Rightarrow \tilde{X}$ be the fuzzy eigen vector corresponding to the fuzzy eigenvalue $\lambda$ for a fuzzy matrix $\tilde{A}$.

Hence the fuzzy eigenvector $\tilde{X}$ of an fuzzy matrix $\tilde{A}$ is not unique.

Property 8 :

If $\tilde{A}$ and $\tilde{B}$ are two fuzzy matrices of order n and $\tilde{B}$ is a non-singular fuzzy matrix then $\tilde{A}$ and $\tilde{B}^{-1}\tilde{A}\tilde{B}$ have same fuzzy eigenvalues.

Proof :

Characteristic polynomial of $\tilde{B}^{-1}\tilde{A}\tilde{B}=\det(\tilde{B}^{-1}\tilde{A}\tilde{B}-\lambda I)=\det(\tilde{B}^{-1}\tilde{A}\tilde{B}-\tilde{B}^{-1}(\lambda I)\tilde{B})=\det(\tilde{B}^{-1}(\tilde{A}-\lambda I B=\det B-1 det A-\lambda I \det B=\det B-1 \det B det A-\lambda I=\det B-1 B det A-\lambda I=\det I det A-\lambda I=det A-\lambda I=$ Characteristic polynomial of $\tilde{A}$. Hence $\tilde{A}$ and $\tilde{B}^{-1}\tilde{A}\tilde{B}$ have same fuzzy eigenvalues.

## E. Conclusion :

In this work, I mainly investigate some properties of fuzzy eigenvectors and fuzzy eigenvalues and give a constructive idea for the same. In the section 'A' I introduce the topics, section 'B' contains some definitions to understand the paper easily, section 'C' contains the constructive idea for fuzzy eigenvalues and corresponding fuzzy eigenvector which are the central focus of this paper, section 'D' contains some properties and lastly section 'E' contains the conclusion.

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